Acta Crystallographica Section A
Foundations of Crystallography
ISSN 0108-7673

Received 19 December 2008
Accepted 27 April 2009

To the memory of Zdeňka.

On a minimum tetrahedron in a three-dimensional lattice. Part I. Lattices with a shortest basis fulfilling $\mathbf{b}\cdot\mathbf{c} \geq 0$, $\mathbf{a}\cdot\mathbf{c} \geq 0$, $\mathbf{a}\cdot\mathbf{b} \geq 0$

B. Gruber‡

Sochařská 14, 170 00 Prague 7, Czech Republic. Correspondence e-mail: magda.fortova@tiscali.cz

The problem of a representative body of a three-dimensional lattice is considered. The cell fulfilling $a + b + c = \min$ is clearly not unique: even five mutually non-congruent such cells can exist in some lattices [Gruber (1973). *Acta Cryst.* A**29**, 433–440]. The idea that this number could be reduced by replacing the cell (*i.e.* a parallelepiped) by another, possibly more suitable, geometrical object is considered. For this object a lattice tetrahedron fulfilling the condition $a + b + c + d + e + f = \min$ is chosen, a to f being the lengths of its edges. It is called the *minitetrahedron* of the lattice. In this article, the problem is solved in detail for lattices that can be generated by a basis **a**, **b**, **c** fulfilling $|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| = \min$, $\mathbf{b}\cdot\mathbf{c} \geq 0$, $\mathbf{a}\cdot\mathbf{c} \geq 0$, $\mathbf{a}\cdot\mathbf{b} \geq 0$. It turns out that in this case not more than two mutually non-congruent minitetrahedra can exist. Necessary and sufficient conditions for the uniqueness are found. They have the form of inequalities between the lengths of the edges and diagonals of the parallelepiped formed by the vectors **a**, **b**, **c**. A procedure for determining all minitetrahedra of a given lattice is shown. Some results are illustrated graphically and all assertions are proved mathematically.

1. Crystallographic kernel

1.1. Introduction

This paper was incited by the problem of a unique representation of a lattice. For this purpose the so-called Niggli cell is usually used. It is characterized by the condition

$$a + b + c = \min \tag{1}$$

and a system, say **S**, of rather complicated inequalities originating in algebra (Eisenstein, 1851; Niggli, 1928; *International Tables for Crystallography*, 2002). Condition (1) is simple and very natural; however, it does not guarantee the uniqueness of the cell: as many as five different cells with this property may exist in some lattices (Gruber, 1973). For this purpose there serves the system **S** which chooses from the cells selected by the minimum condition (1) the final unique cell.

Thus the procedure consists of two steps. This indirectness is not exactly welcome. *One of the aims of this paper is to lower the number of bodies admitted by the minimum condition* from which the representative body originates.

Our idea lies in the observation that not all eight vertices of a primitive cell are necessary for determining the lattice: only four of them suffice unless they lie in a plane. Thus a tetrahedron is equally justified as a representative body of a lattice. Now a tetrahedron can be completed in four different ways into a cell that determines the same lattice as the tetrahedron. Thus to give one tetrahedron means the same as to give four cells: the tetrahedron covers all of them. This suggests a way of lowering the number of bodies admitted by the minimum condition: instead of with cells with the shortest edges we shall work with tetrahedra with the shortest edges.

The tetrahedra may also be of interest from other aspects. For example, to define a lattice by means of a tetrahedron means to define it in a 'homogeneous' way, that is by six straight segments instead of by three straight segments and three angles as for the cells. Moreover, the relations between the shape of the minimum tetrahedron and, *e.g.*, the Bravais type of the lattice may reveal interesting information.

From the mathematical point of view in the 'cell case' we are looking for three shortest lattice vectors on the condition that they are linearly independent, whereas in the 'tetrahedron case' we are looking for six shortest lattice vectors on the condition that they form a tetrahedron. In the latter case we can therefore expect greater formal complications. This proved to be true.

Finally the author must confess that he became during the work more and more fascinated by the tetrahedron as a body itself. After all, it is the simplest possible three-dimensional body being determined by the smallest possible number of quite arbitrary points.

‡ Retired. Formerly at the Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic.

1.2. Notions, conventions and definitions

We say that the tetrahedron T is a *lattice tetrahedron* of the lattice L if its vertices are lattice points of L. Let the edges of a tetrahedron T be denoted $e_1, ..., e_6$. Then we define

$$\sigma(T) := \sum_{i=1}^{6} \overline{e}_i,$$

where $\overline{e}_i$ means the length of the edge e_i. More generally we define

$$\sigma_\tau(T) := \sum_{i=1}^{6} \overline{e}_i^\tau$$

for any real $\tau > 0$.

Definition 1.1. We say that a lattice tetrahedron T of the lattice L is a *minitetrahedron* of L if $\sigma(T) \leq \sigma(T')$ holds for any lattice tetrahedron T' of L.

The set of all minitetrahedra of L is denoted $\mathbf{M}$. On this set $\mathbf{M}$ we define a decomposition $\mathbf{D}$ into classes of mutually congruent minitetrahedra. We introduce the following formulations.

Definitions 1.2.

(i) We say that 'the minitetrahedron of the lattice L is *unique*' if $\mathbf{D}$ consists of one class only.

(ii) In the opposite case we say that 'the minitetrahedron of L is *ambiguous*'.

(iii) In particular we say that 'the minitetrahedron of L is k-times ($k > 1$) ambiguous' if $\mathbf{D}$ consists exactly of k classes.

Our main concern in this paper is the uniqueness or ambiguity of the minitetrahedron of a given lattice.

If $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are linearly independent lattice vectors of the lattice L then the symbol $\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle$ denotes the set of all tetrahedra with the vertices

$$O, \quad O + \delta\mathbf{p}, \quad O + \delta\mathbf{q}, \quad O + \delta\mathbf{r}$$

where $|\delta| = 1$ and O ranges over all lattice points of L. We call such a set an *abstract tetrahedron*. The symbol $\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle$ is not unique; for example

$$\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle = \langle \mathbf{q}, -\mathbf{p} + \mathbf{q}, \mathbf{q} - \mathbf{r} \rangle$$

etc. We usually choose the symbol that seems to us the simplest.

Since all tetrahedra T from an abstract tetrahedron $\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle$ are congruent, we can transfer many properties of T directly to this abstract tetrahedron. We shall, *e.g.*, speak about the congruence between a tetrahedron T and an abstract tetrahedron $\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle$ *etc*. If $\langle \mathbf{p}, \mathbf{q}, \mathbf{r} \rangle$ contains a minitetrahedron then it is a part of a class of the decomposition $\mathbf{D}$.

It remains to choose the starting point of our investigations and the frame in terms of which our findings will be stated. It seems natural to choose for this purpose a basis with the shortest vectors, that is a basis $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ fulfilling

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| \leq |\mathbf{a}'| + |\mathbf{b}'| + |\mathbf{c}'|$$

for any basis $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ of L. Let us call it a *minimum basis*. It can be transformed – if necessary – by a mere change of direction of one of its vectors into such a form that either

$$\mathbf{b} \cdot \mathbf{c} \geq 0, \quad \mathbf{a} \cdot \mathbf{c} \geq 0, \quad \mathbf{a} \cdot \mathbf{b} \geq 0 \tag{2}$$

or

$$\mathbf{b} \cdot \mathbf{c} < 0, \quad \mathbf{a} \cdot \mathbf{c} < 0, \quad \mathbf{a} \cdot \mathbf{b} < 0. \tag{3}$$

In this paper, with the subtitle Part I, we confine ourselves solely (with the exception of §3) to lattices that have at least one minimum basis fulfilling (2), leaving the remaining lattices to Part II. This division is not only formal. The two cases differ deeply not only in their results but also in the methods and the whole geometrical image.

To simplify things we further normalize the notation to

$$|\mathbf{a}| \leq |\mathbf{b}| \leq |\mathbf{c}|. \tag{4}$$

The first question now is how to recognize whether a basis $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ fulfilling (2) and (4) is a minimum basis. This occurs if and only if the inequalities

$$2\mathbf{b} \cdot \mathbf{c} \leq \mathbf{b}^2, \quad 2\mathbf{a} \cdot \mathbf{c} \leq \mathbf{a}^2, \quad 2\mathbf{a} \cdot \mathbf{b} \leq \mathbf{a}^2 \tag{5}$$

hold (see *e.g.* Gruber, 1997).

The second question is how to find such a basis (if it exists) when an arbitrary basis of the lattice L is known. Here the algorithm by Křivý & Gruber (1976) (K&G), especially in the form given by Gruber (1997), can be applied. Since our normalization (2) and (3) differs from that used in *International Tables for Crystallography* (IT) and the algorithm by K&G[1] the answer is as follows:

Proposition 1.3. Let $\mathbf{a}_0$, $\mathbf{b}_0$, $\mathbf{c}_0$ be the vectors in the output of the K&G algorithm. Then the following is true:

(i) The results of the present paper are applicable to the lattice if and only if

$$(\mathbf{b}_0 \cdot \mathbf{c}_0)\,(\mathbf{a}_0 \cdot \mathbf{c}_0)\,(\mathbf{a}_0 \cdot \mathbf{b}_0) \geq 0.$$

{The crux is this. There exist lattices with two minimum bases, one fulfilling (2) and the other (3). [*E.g.* the lattice generated by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ which obey $a = b = c = 4\mathbf{b} \cdot \mathbf{c} = 2\mathbf{a} \cdot \mathbf{c} = 2\mathbf{a} \cdot \mathbf{b} = 1$ has – besides the minimum basis $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ fulfilling (2) – also a minimum basis $\mathbf{a}' = \mathbf{b}$, $\mathbf{b}' = \mathbf{a} - \mathbf{b}$, $\mathbf{c}' = \mathbf{c}$ with $\mathbf{b}' \cdot \mathbf{c}' < 0$, $\mathbf{a}' \cdot \mathbf{c}' < 0$, $\mathbf{a}' \cdot \mathbf{b}' < 0$.] Then we have to show that the basis $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ cannot appear in the output of the K&G algorithm. This follows from a detailed analysis in Gruber (1978).}

(ii) If it is so, at least one of the triplets

$$\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0, \quad -\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0, \quad \mathbf{a}_0, -\mathbf{b}_0, \mathbf{c}_0, \quad \mathbf{a}_0, \mathbf{b}_0, -\mathbf{c}_0$$

– re-denoted as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ – fulfils the inequalities (2), (4) and (5).

Let us conclude this section by the following

General assumption 1.4. From this point on until the end of the paper (apart from §3) *it is assumed that L is a three-dimensional lattice and* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *is one of its bases fulfilling*

[1] This is not negligence on the part of the author. Following the IT convention we would unnecessarily lose the lattices with $(\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b}) = 0$.

Table 1
Uniqueness, ambiguity and shapes of the minitetrahedra.

The symbol # separates those that are not congruent.

Conditions	Shapes
$\|\mathbf{a}-\mathbf{b}\| > \|\mathbf{a}+\mathbf{b}-\mathbf{c}\|$	$\langle \mathbf{a}, \mathbf{c}, -\mathbf{b}+\mathbf{c}\rangle$
$\|\mathbf{a}-\mathbf{b}\| = \|\mathbf{a}+\mathbf{b}-\mathbf{c}\|$, $\alpha > 60^\circ$, $\beta > 60^\circ$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ # $\langle \mathbf{a}, \mathbf{c}, -\mathbf{b}+\mathbf{c}\rangle$
$\|\mathbf{a}-\mathbf{b}\| = \|\mathbf{a}+\mathbf{b}-\mathbf{c}\|$, either $\alpha \le 60^\circ$ or $\beta \le 60^\circ$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$
$\|\mathbf{a}-\mathbf{c}\| > \|\mathbf{a}-\mathbf{b}+\mathbf{c}\|$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{b}-\mathbf{c}\rangle$
$\|\mathbf{a}-\mathbf{c}\| = \|\mathbf{a}-\mathbf{b}+\mathbf{c}\|$, $\|\mathbf{c}\| < \|\mathbf{b}-\mathbf{c}\|$, $\gamma > 60^\circ$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ # $\langle \mathbf{a}, \mathbf{b}, \mathbf{b}-\mathbf{c}\rangle$
$\|\mathbf{a}-\mathbf{c}\| = \|\mathbf{a}-\mathbf{b}+\mathbf{c}\|$, either $\|\mathbf{c}\| \ge \|\mathbf{b}-\mathbf{c}\|$ or $\gamma \le 60^\circ$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$
$\|\mathbf{b}-\mathbf{c}\| > \|-\mathbf{a}+\mathbf{b}+\mathbf{c}\|$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{a}-\mathbf{c}\rangle$
$\|\mathbf{b}-\mathbf{c}\| = \|-\mathbf{a}+\mathbf{b}+\mathbf{c}\|$, $\|\mathbf{b}\| < \|\mathbf{a}-\mathbf{b}\|$, $\|\mathbf{c}\| < \|\mathbf{a}-\mathbf{c}\|$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ # $\langle \mathbf{a}, \mathbf{b}, \mathbf{a}-\mathbf{c}\rangle$
$\|\mathbf{b}-\mathbf{c}\| = \|-\mathbf{a}+\mathbf{b}+\mathbf{c}\|$, either $\|\mathbf{b}\| \ge \|\mathbf{a}-\mathbf{b}\|$ or $\|\mathbf{c}\| \ge \|\mathbf{a}-\mathbf{c}\|$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$
$\|\mathbf{a}-\mathbf{b}\| < \|\mathbf{a}+\mathbf{b}-\mathbf{c}\|$, $\|\mathbf{a}-\mathbf{c}\| < \|\mathbf{a}-\mathbf{b}+\mathbf{c}\|$, $\|\mathbf{b}-\mathbf{c}\| < \|-\mathbf{a}+\mathbf{b}+\mathbf{c}\|$	$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$

$$|\mathbf{a}| \le |\mathbf{b}| \le |\mathbf{c}|,$$
$$0 \le 2\mathbf{b}\cdot\mathbf{c} \le \mathbf{b}^2, \quad 0 \le 2\mathbf{a}\cdot\mathbf{c} \le \mathbf{a}^2, \quad 0 \le 2\mathbf{a}\cdot\mathbf{b} \le \mathbf{a}^2.$$

This convention will be used throughout the paper without the reader being reminded.

1.3. Main results

Theorem 1.5. The minitetrahedron of the lattice L is either unique or twice ambiguous.

Theorem 1.6. The minitetrahedron of the lattice L is twice ambiguous if and only if one of the three following conditions is fulfilled:

$$\begin{array}{llll} \text{(i)} & |\mathbf{a}-\mathbf{b}| = |\mathbf{a}+\mathbf{b}-\mathbf{c}|, & \alpha > 60^\circ, & \beta > 60^\circ, \\ \text{(ii)} & |\mathbf{a}-\mathbf{c}| = |\mathbf{a}-\mathbf{b}+\mathbf{c}|, & |\mathbf{c}| < |\mathbf{b}-\mathbf{c}|, & \gamma > 60^\circ, \\ \text{(iii)} & |\mathbf{b}-\mathbf{c}| = |-\mathbf{a}+\mathbf{b}+\mathbf{c}|, & |\mathbf{b}| < |\mathbf{a}-\mathbf{b}|, & |\mathbf{c}| < |\mathbf{a}-\mathbf{c}|. \end{array}$$

The particulars are given in Table 1.

These two theorems are the kernel of this paper. The second theorem is illustrated in Fig. 1. From Table 1 two easy consequences follow.

Proposition 1.7. If T, T' are two non-congruent minitetrahedra of L, then one of them is congruent with $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$.

Proposition 1.8. If T is a minitetrahedron of L then it is congruent with one of the following four abstract tetrahedra:

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle, \quad \langle \mathbf{a}, \mathbf{b}, \mathbf{a}-\mathbf{c}\rangle, \quad \langle \mathbf{a}, \mathbf{b}, \mathbf{b}-\mathbf{c}\rangle, \quad \langle \mathbf{a}, \mathbf{c}, -\mathbf{b}+\mathbf{c}\rangle.$$

Thus to find the shapes of all minitetrahedra of L it is sufficient to compare the shapes of four lattice tetrahedra (Fig. 2). (For this purpose it may be advantageous to use Theorem 1.10 with $\tau = 2$.)

Theorem 1.9. Let T and T' be minitetrahedra of L. Then their edges may be denoted

$$e_1, \ldots, e_6 \ \text{and} \ e'_1, \ldots, e'_6$$

in such a way that

$$\overline{e}_i = \overline{e'_i} \ \text{for} \ i = 1, \ldots, 6.$$

Thus two minitetrahedra of L agree not only in the *sum* of lengths of their edges but also in the lengths of *individual* edges.

Theorem 1.10. Let T be a lattice tetrahedron of the lattice L and $\tau > 0$ a real number. Then T is a minitetrahedron of L if and only if

$$\sigma_\tau(T) \le \sigma_\tau(T')$$

is true for any lattice tetrahedron T'.

Example 1.11. Let the lattice L be generated by the vectors **a**, **b**, **c** fulfilling

$$\begin{pmatrix} \mathbf{a}^2 & \mathbf{b}^2 & \mathbf{c}^2 \\ 2\mathbf{b}\cdot\mathbf{c} & 2\mathbf{a}\cdot\mathbf{c} & 2\mathbf{a}\cdot\mathbf{b} \end{pmatrix} = \begin{pmatrix} 3 & 4 & 4 \\ 4 & 1 & 3 \end{pmatrix}.$$

Then the condition (1) offers five mutually non-congruent 'minicells':[2]

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}], \quad [\mathbf{a}, \mathbf{b}, \mathbf{b}-\mathbf{c}], \quad [\mathbf{a}, -\mathbf{c}, -\mathbf{a}+\mathbf{b}],$$
$$[\mathbf{a}, -\mathbf{c}, -\mathbf{b}+\mathbf{c}], \quad [\mathbf{a}, -\mathbf{a}+\mathbf{b}, -\mathbf{b}+\mathbf{c}],$$

whereas the condition

$$a + b + c + d + e + f = \min \tag{6}$$

leads directly to the unique minitetrahedron $\langle \mathbf{a}, \mathbf{b}, \mathbf{b}-\mathbf{c}\rangle$ (according to Assumption 1.4, Theorem 1.6 and the fourth row of Table 1).

Theorem 1.12. Let the lattice L be of the Bravais type xY. Then the following is true:

(i) If xY is one of the Bravais types

$$cP,\ cF,\ tP,\ oP,\ oC,\ hP,\ mP$$

then the minitetrahedron of L is unique.

(ii) If xY is one of the Bravais types

$$oI,\ mI,\ aP$$

then there exists a lattice of the type xY which has an ambiguous minitetrahedron (see the following Examples 1.13).

(iii) In the remaining cases

$$cI,\ tI,\ oF,\ hR$$

the decision cannot yet be made (*i.e.* before Part II is completed).

Examples 1.13. Let the lattice L be given by its body-centred cell $[\mathbf{a}', \mathbf{b}', \mathbf{c}']$ fulfilling

$$a' = 3, \ \ b' = 4, \ \ c' = 5\sqrt{3}, \ \ \alpha' = \beta' = \gamma' = 90^\circ$$
$$(\text{or} \ \ a' = 2, \ \ b' = c' = \sqrt{7}, \ \ \alpha' = \gamma' = 90^\circ, \ \ 2\sqrt{7}\cos\beta' = -1).$$

[2] From these cells the unique Niggli cell must be selected by Eisenstein's rules [see *e.g.* IT (2002), §9.2.2].

Then L belongs to the Bravais type oI (or mI) [being 'accidentally' not of a higher symmetry; see Table 9.3.4.1 in IT (2002)] and its minitetrahedron is ambiguous.

Hint. Put

$$\mathbf{a} := \mathbf{a}', \quad \mathbf{b} := \mathbf{b}', \quad 2\mathbf{c} := \mathbf{a}' + \mathbf{b}' + \mathbf{c}'$$
$$(\text{or} \ \ \mathbf{a} := \mathbf{a}', \quad 2\mathbf{b} := \mathbf{a}' + \mathbf{b}' + \mathbf{c}', \quad 2\mathbf{c} := \mathbf{a}' - \mathbf{b}' + \mathbf{c}')$$

and construct the minitetrahedra

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle \ \text{ and } \ \langle \mathbf{a}, \mathbf{c}, -\mathbf{b} + \mathbf{c} \rangle$$
$$(\text{or} \ \ \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle \ \text{ and } \ \langle \mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{c} \rangle).$$

2. Proofs

2.1. Further notions and notations

Notation 2.1. The following notations will be used from now on:

$$A := \mathbf{a}^2, \quad B := \mathbf{b}^2, \quad C := \mathbf{c}^2,$$
$$x := 2\mathbf{b} \cdot \mathbf{c}, \quad y := 2\mathbf{a} \cdot \mathbf{c}, \quad z := 2\mathbf{a} \cdot \mathbf{b},$$
$$\Xi := [x, y, z],$$
$$\Omega := \{[u, v, w]; \ \ 0 \le u \le B, \ \ 0 \le v \le A, \ \ 0 \le w \le A\},$$
$$r := x/A, \quad s := y/A, \quad t := z/A,$$
$$\kappa := B/A, \quad \lambda := C/A,$$
$$V := m^2 + \kappa n^2 + \lambda p^2 + rnp + smp + tmn + \omega,$$
$$(m, n, p \text{ integers}, \ \ \omega \text{ real}).$$

Consequently the following relations hold:

$$0 < A \le B \le C,$$
$$0 \le x \le B, \quad 0 \le y \le A, \quad 0 \le z \le A,$$
$$0 \le r \le \kappa, \quad 0 \le s \le 1, \quad 0 \le t \le 1,$$
$$1 \le \kappa \le \lambda,$$
$$\Xi = A[r, s, t].$$

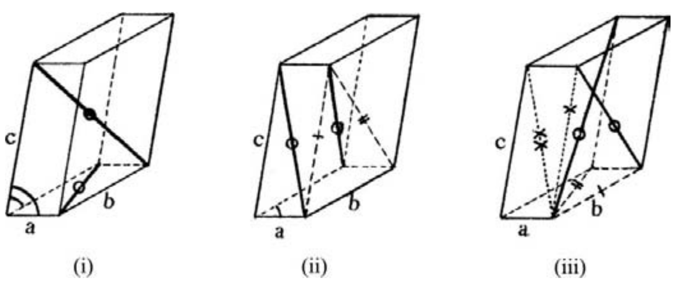


Figure 1
Theorem 1.6 illustrated. The straight lines marked by open circles are of the same length. The straight lines crossed by one short line, |, are shorter than the straight lines crossed by a pair of short lines, ‖. Similarly, the straight lines marked by a cross are shorter than the straight lines marked by a pair of crosses. The angles that are explicitly marked are greater than 60°.

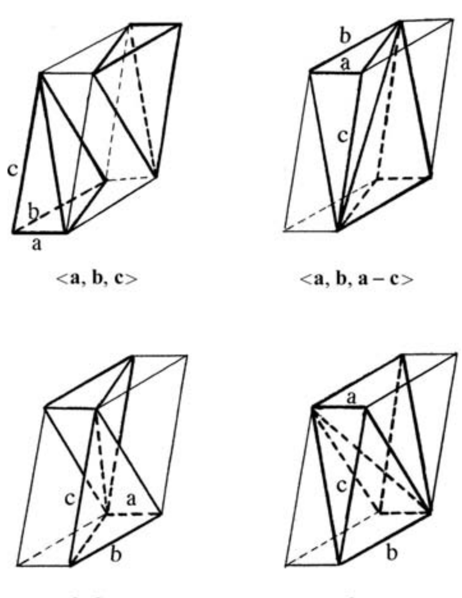


Figure 2
Possible shapes of minitetrahedra.

Definition 2.2. Let

$$\mathbf{s} := m\mathbf{a} + n\mathbf{b} + p\mathbf{c} \tag{7}$$

be a lattice vector of the lattice L.

(i) We say that $\mathbf{s}$ is

an a-vector if $m \ge 1, \ n = 0, \ p = 0$;
a b-vector if $n \ge 1, \ p = 0$;
a c-vector if $p \ge 1$.

(ii) If $\mathbf{s}$ is a c-vector it is said to be
a *red* c-vector if $n = 0$, $p = 1$; and to be
a *green* c-vector if either $n \ne 0$ or $p > 1$ (all with respect to the basis $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$).

Thus the red c-vectors and the green c-vectors form a division of the c-vectors into two classes.

Notation 2.3.

(i) The set of all green c-vectors (7) is denoted $\mathbf{G}$.

(ii) The set of all green c-vectors (7) fulfilling max $(|m|, |n|, |p|) \ge 2$ is denoted $\mathbf{G}_2$.

(iii) In an analogous way the symbols

$$\mathbf{A}, \mathbf{A}_2, \ \ \mathbf{B}, \mathbf{B}_2, \ \ \mathbf{C}, \mathbf{C}_2 \text{ and } \mathbf{R}, \mathbf{R}_2$$

are defined for the

$$a\text{-}, b\text{-}, c\text{- and red } c\text{-vectors.} \tag{8}$$

Definition 2.4. If all green c-vectors are arranged into a sequence

$$\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \ldots \tag{9}$$

in such a way that

(i) any green c-vector occurs in sequence (9) only once and

(ii)

$$|\mathbf{g}_1| \le |\mathbf{g}_2| \le |\mathbf{g}_3| \le \ldots \tag{10}$$

Table 2
Brief notation of some lattice vectors.

Symbol	Lattice vectors
31	$\mathbf{a} + \mathbf{b}$
32	$-\mathbf{a} + \mathbf{b}$
33	$\mathbf{a} + \mathbf{c}$
34	$-\mathbf{a} + \mathbf{c}$
35	$\mathbf{b} + \mathbf{c}$
36	$-\mathbf{b} + \mathbf{c}$
37	$\mathbf{a} + \mathbf{b} + \mathbf{c}$
38	$-\mathbf{a} + \mathbf{b} + \mathbf{c}$
39	$\mathbf{a} - \mathbf{b} + \mathbf{c}$
40	$-\mathbf{a} - \mathbf{b} + \mathbf{c}$

we say that (9) is a *normal sequence* of green c-vectors of the lattice L. In a similar way normal sequences of the vectors in (8) are introduced.

Remarks 2.5.
(i) The sequence (9) is generally not unique, unlike the sequence (10).
(ii) If

$$n \geq 1, \quad 1 \leq k_1 < k_2 < \ldots < k_n$$

are integers and (9) is a normal sequence of green c-vectors then

$$|\mathbf{g}_n| \leq |\mathbf{g}_{k_n}|,$$
$$|\mathbf{g}_1| + \ldots + |\mathbf{g}_n| \leq |\mathbf{g}_{k_1}| + \ldots + |\mathbf{g}_{k_n}|.$$

Similarly for the other normal sequences.

Notations 2.6. For the sake of brevity we make the following conventions:
(i) Instead of

$$|\mathbf{s}| = |\mathbf{t}|, \quad |\mathbf{s}| < |\mathbf{t}|$$

we shall also write

$$\mathbf{s} \doteq \mathbf{t}, \quad \mathbf{s} \mathrel{\dot{<}} \mathbf{t}.$$

(ii) Some frequently appearing lattice vectors will sometimes be denoted by bold numerals **31**, ..., **40** according to Table 2.

Although this way may look rather unusual, in the end it proved quite apt. For example (see the second entry in Table 4)

$$\mathbf{b} \mathrel{\dot{<}} \mathbf{31} \doteq \mathbf{32} \mathrel{\dot{<}} \ldots \tag{11}$$

will mean that

$$|\mathbf{b}| < |\mathbf{a} + \mathbf{b}| = |-\mathbf{a} + \mathbf{b}| < |\mathbf{s}|$$

for any b-vector $\mathbf{s}$ different from the vectors $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$, $-\mathbf{a} + \mathbf{b}$. Another example:

$$\mathbf{G} = \mathbf{G}_2 \cup \{\mathbf{35}, \ \mathbf{36}, \ \mathbf{37}, \ \mathbf{38}, \ \mathbf{39}, \ \mathbf{40}\}.$$

Definition 2.7. Fourteen particular points will be of special importance. They will usually be referred to as *selected* points. Their notation and coordinates are given in Table 3.

Table 3
Notation and coordinates of selected points.

Notation	Coordinates	Condition
F	$[A/2, A, A]$	
H	$[B, A/2, A]$	
J	$[B, A, A]$	
M	$[B, A, (A + B - C)/2]$	$(C < A + B)$
N	$[C - A, A, 0]$	$(A < C < A + B)$
O	$[0, 0, 0]$	
P	$[B, C - B, 0]$	$(B < C < A + B)$
Γ	$[0, 0, A]$	
Δ	$[0, A, A]$	
Θ	$[B, 0, A]$	
Λ	$[0, A, 0]$	
Σ	$[B, 0, 0]$	
Φ	$[B - A, 0, A]$	$(A < B)$
Ψ	$[B, A, 0]$	

They can be seen in Figs. 3, 4, 5 and 6. Eight of them, namely

$$J, O, \Gamma, \Delta, \Theta, \Lambda, \Sigma, \Psi$$

are vertices of the parallelepiped Ω. The reasons for considering the remaining six points

$$F, H, M, N, P \tag{12}$$

as selected points are suggested in the following.

Intermezzo 2.8. General scheme of the proof. (Here we use informal language.) It does not seem a bad idea to look for the minitetrahedra among those lattice tetrahedra that have in their edges all three vectors **a**, **b**, **c**, since these vectors are 'as short as possible'. We call such tetrahedra the *auxiliary* tetrahedra (see further Definition 2.25, Lemma 2.26 and Table 7).

First let us suppose that the tetrahedron $T \in \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a minitetrahedron of L. (This occurs, *e.g.*, if $\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = 0$.) Then, in particular,

$$\sigma(T) \leq \sigma(T') \tag{13}$$

for any *auxiliary* tetrahedron T'. Then the inequality (13) means

$$|\mathbf{a} - \mathbf{b}| + |\mathbf{a} - \mathbf{c}| + |\mathbf{b} - \mathbf{c}| \leq |\mathbf{q}_1| + |\mathbf{q}_2| + |\mathbf{q}_3|$$

where $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ are linear expressions in **a**, **b**, **c**. This can be written

$$\sqrt{\mathrm{P}_1} + \sqrt{\mathrm{P}_2} + \sqrt{\mathrm{P}_3} \leq \sqrt{\mathrm{Q}_1} + \sqrt{\mathrm{Q}_2} + \sqrt{\mathrm{Q}_3}, \tag{14}$$

where P_i, Q_j are linear expressions in x, y, z. To deal with such a complicated inequality generally is hopeless. However, there exist three auxiliary abstract tetrahedra, namely[3]

$$\mathbf{T}_1 := \langle \mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{c} \rangle, \quad \mathbf{T}_2 := \langle \mathbf{a}, \mathbf{b}, \mathbf{b} - \mathbf{c} \rangle,$$
$$\mathbf{T}_3 := \langle \mathbf{a}, \mathbf{c}, -\mathbf{b} + \mathbf{c} \rangle$$

such that for $T \in \mathbf{T}_i$ $(1 \leq i \leq 3)$ the inequality (14) reduces to

$$\sqrt{P_r} \leq \sqrt{Q_s}, \tag{15}$$

[3] In fact, according to Table 7 $\mathbf{T}_1 = \mathbf{\mathit{56}}$, $\mathbf{T}_2 = \mathbf{\mathit{59}}$, $\mathbf{T}_3 = \mathbf{\mathit{63}}$.

the numbers r, s being dependent on i. This inequality can be written

$$L_i \leq 0, \tag{16}$$

where L_i is a linear function in x, y, z. The intersection of the plane $L_i = 0$ with the parallelepiped Ω is mostly[4] a triangle whose vertices are selected points (see Figs. 6, 3, 4 and 5). Finally denote Ω_i ($1 \leq i \leq 3$) the set of points from Ω for which $L_i \geq 0$ and Ω_0 the set of those points from Ω for which $L_1 \leq 0$, $L_2 \leq 0$, $L_3 \leq 0$.

Summarizing we can say: *If* $T \in \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ *is a minitetrahedron of* L *then* $\Xi \in \Omega_0$. As we shall show, this assertion can be inverted: *If* $\Xi \in \Omega_0$ *then* $T \in \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ *is a minitetrahedron*; that is, the inequality (13) holds not only for $T' \in \mathbf{T}_i$ ($1 \leq i \leq 3$) but for any lattice tetrahedron T' of L.

But this is not enough: *If* $\Xi \in \Omega_i$ ($i = 1, 2, 3$) *then* $T \in \mathbf{T}_i$ *is a minitetrahedron* of L. This is much more than we were justified to expect. All these assertions, of course, have to be rigorously proved.

Viewing the proof as a whole we can distinguish five steps:

(i) To find – for any point $\Xi \in \Omega$ – all suitable 'as short as possible' lattice vectors in a number sufficient for constructing all minitetrahedra (§§2.3 and 2.4).

(ii) To ascertain all possible distributions of the *a*-, *b*- and *c*-vectors along the edges of an arbitrary lattice tetrahedron (§2.5).

(iii) To find – for any $\Xi \in \Omega$ – *one* minitetrahedron (§2.6).

(iv) To find – for any $\Xi \in \Omega$ – *all* minitetrahedra and divide them into classes of mutually congruent ones (§2.7).

(v) According to the number of these classes to decide between the uniqueness and ambiguity of the minitetrahedron of the lattice L (§2.8).

2.2. Auxiliary inequalities

Proposition 2.9. Let $a \neq 0$, $K > 0$, $\xi_1 < \xi_2$, denote

$$P(\xi) := a\xi^2 + b\xi + c, \quad D := b^2 - 4ac, \quad E := 2|a|K - |b|.$$

[The expression D is usually called the discriminant of the polynomial $P(\xi)$.] Then the following is true:

(i) if $D < 0$ then

$$(\operatorname{sgn} a)\, P(\xi) > 0 \tag{17}$$

for any ξ;

(ii) if $\operatorname{sgn} P(\xi_1) = \operatorname{sgn} P(\xi_2) = -\operatorname{sgn} a$ then

$$\operatorname{sgn} P(\xi) = \operatorname{sgn} P(\xi_1)$$

for any ξ fulfilling $\xi_1 \leq \xi \leq \xi_2$;

(iii) if $D \geq 0$, $E \geq 0$, $E^2 > D$ then (17) holds for $|\xi| \geq K$.

Proof. Points (i) and (ii) are known from algebra. In point (iii) assume first $a > 0$. We have to prove that the zero points of the polynomial $P(\xi)$ lie in the open interval $(-K, K)$. This means

$$-2aK < -b - \sqrt{D}, \quad -b + \sqrt{D} < 2aK,$$

which can be written

$$\sqrt{D} < 2aK - |b| = E.$$

This is equivalent to $D < E^2$ for $E \geq 0$. Secondly, if $a < 0$ we apply what has been just proved to the polynomial $-P(\xi)$.

Proposition 2.10. Let $p \geq 2$, denote

$$\nu := t^2 - 4\kappa,$$
$$Q := \nu\omega + 4(r^2 + s^2\kappa - rst + \nu\lambda).$$

Then the following is true: If

$$\nu < 0, \quad \omega < 0, \quad Q < 0, \tag{18}$$

then

$$V > 0. \tag{19}$$

Proof. Denote $W := r^2 + s^2\kappa - rst + \nu\lambda$ so that $Q = \nu\omega + 4W$. From $Q < 0$, $\nu\omega > 0$ it follows $W < 0$. The inequality $Q < 0$ means $4 > -\nu\omega/W$ so that also $p^2 > -\nu\omega/W$ and consequently

$$U := p^2 W + \nu\omega < 0. \tag{20}$$

But $16U$ is the discriminant of the expression

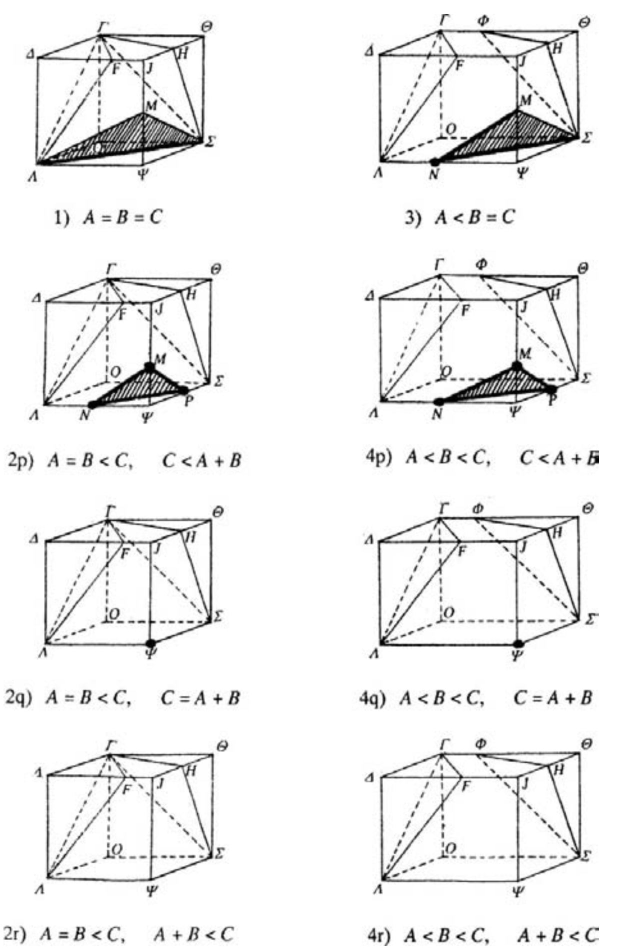


Figure 3
The set of all ambiguity points for $|\mathbf{a} - \mathbf{b}| = |\mathbf{a} + \mathbf{b} - \mathbf{c}|$, $\alpha > 60°$, $\beta > 60°$.

[4] *I.e.* for $i = 1, 2$ and $i = 3$, $C < A + B$. In the remaining alternatives it is either the point Ψ or the empty set.

$$R := vn^2 + 2pn(st - 2r) + s^2p^2 - 4\lambda p^2 - 4\omega$$

taken as a quadratic function of the variable n. Thus from (20) and $v < 0$ it follows that $R < 0$ according to Proposition 2.9 point (i). However, R itself is the discriminant of the expression V taken as a quadratic function of m. Applying Proposition 2.9 (i) once more we get finally (19).

Proposition 2.11. Let $p = 1$, $|n| \geq 2$, denote

$$v := t^2 - 4\kappa,$$
$$Q := (st - 2r)^2 - v(s^2 - 4\lambda - 4\omega),$$
$$R := |v| - |0.5st - r|$$

and assume

$$v < 0. \tag{21}$$

Then the following is true: if either

$$Q < 0$$

or

$$Q \geq 0, \quad R \geq 0, \quad 4R^2 > Q \tag{22}$$

then

$$V > 0.$$

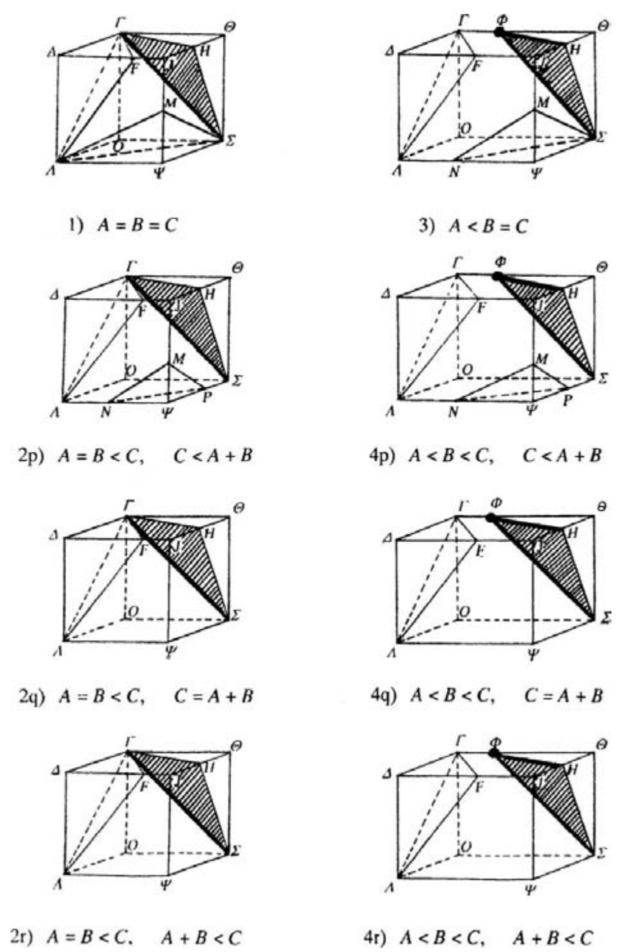


Figure 4
The set of all ambiguity points for $|\mathbf{a} - \mathbf{c}| = |\mathbf{a} - \mathbf{b} + \mathbf{c}|$, $|\mathbf{c}| < |\mathbf{b} - \mathbf{c}|$, $\gamma > 60°$.

Proof. We apply Proposition 2.9 to the expression

$$W := vn^2 + 2(st - 2r)n + s^2 - 4\lambda - 4\omega$$

taken as a function of the variable n. Putting

$$a := v, \quad b := 2(st - 2r), \quad c := s^2 - 4\lambda - 4\omega, \quad K = 2,$$

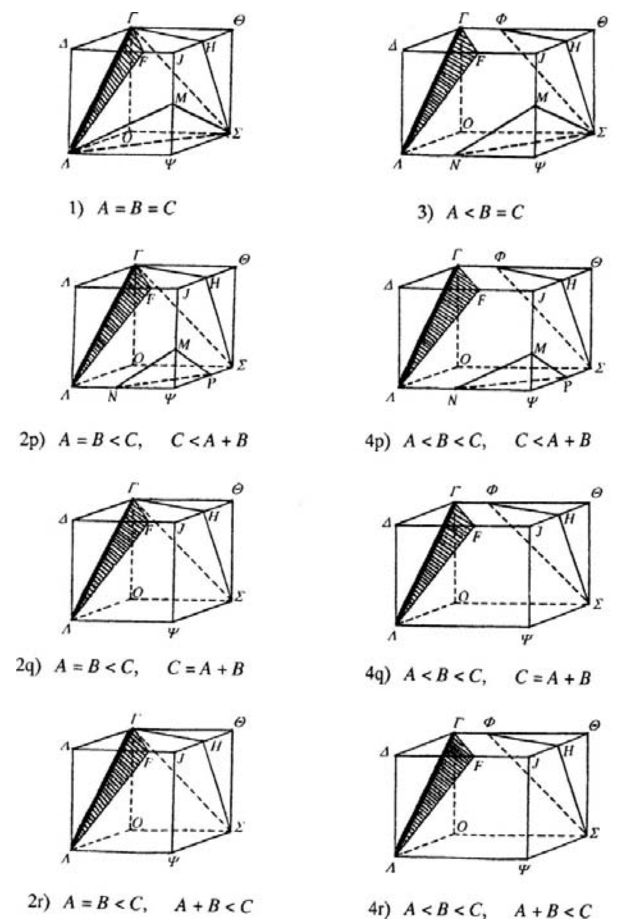


Figure 5
The set of all ambiguity points for $|\mathbf{b} - \mathbf{c}| = |-\mathbf{a} + \mathbf{b} + \mathbf{c}|$, $|\mathbf{b}| < |\mathbf{a} - \mathbf{b}|$, $|\mathbf{c}| < |\mathbf{a} - \mathbf{c}|$.

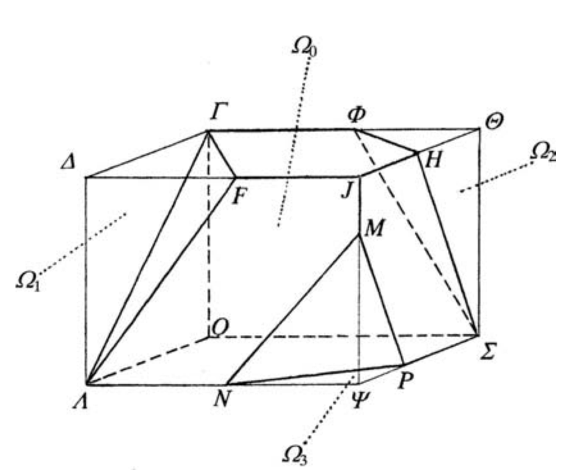


Figure 6
The 'truncated' parallelepiped Ω_0 and the three bodies Ω_1, Ω_2, Ω_3 from Intermezzo 2.8 on the condition $A < B < C < A + B$.

we get

$$D = 4Q, \quad E = 4R$$

and Proposition 2.9 gives

$$W < 0 \text{ for } |n| \geq 2.$$

But W is the discriminant of V (with $p = 1$) taken as a function of m. Thus V has no zero points and is therefore positive. QED

Proposition 2.12. Let $p = 1$, $|n| = 1$, $|m| \geq 2$, denote

$$v := s + tn,$$
$$Q := v^2 - 4(\kappa + \lambda + rn + \omega),$$
$$R := 4 - |v|.$$

Then the following is true: if either

$$Q < 0 \tag{23}$$

or

$$Q \geq 0, \quad R \geq 0, \quad R^2 > Q \tag{24}$$

then

$$V > 0.$$

Proof. Here explicitly

$$V := m^2 + (s + tn)m + \kappa + \lambda + rn + \omega.$$

We apply Proposition 2.9 getting thus $D = Q$, $E = R$.

Definition 2.13.

(i) If $\Xi \in \{F, J, N, O, \Gamma, \Theta, \Lambda, \Phi, \Psi\}$ put $\omega = r - \kappa - \lambda$.
(ii) If $\Xi = \Delta$ put $\omega = -r + s + t - 1 - \kappa - \lambda$.
(iii) If $\Xi \in \{H, \Sigma\}$ put $\omega = r - s + t - 1 - \kappa - \lambda$.
(iv) If $\Xi \in \{M, P\}$ put $\omega = r + s - t - 1 - \kappa - \lambda$.

Proposition 2.14. Let Ξ be one of the 14 selected points of Table 3. Determine ω according to Definition 2.13. Let either

$$p \geq 2 \tag{25}$$

or

$$p = 1, \quad |n| \geq 2 \tag{26}$$

or

$$p = 1, \quad |n| = 1, \quad |m| \geq 2. \tag{27}$$

Then

$$V > 0.$$

Proof. Most complicated is the case $\Xi = M$. We shall therefore perform it in detail. From Table 3 and Definition 2.13 (iv) it follows that

$$r = \kappa, \quad s = 1, \quad t = (\kappa - \lambda + 1)/2, \quad \omega = -(\kappa + \lambda + 1)/2.$$

The condition $C < A + B$ means that we are interested only in $\lambda \in \mathbf{J} := \{\xi; \kappa \leq \xi < \kappa + 1\}$.

First let $p \geq 2$. We want to apply Proposition 2.10 and therefore verify its assumptions (18). We calculate

$$4v(\lambda) = \lambda^2 - 2(\kappa + 1)\lambda + \kappa^2 - 14\kappa + 1, \tag{28}$$

$$8Q(\lambda) = 7\lambda^3 - 15(\kappa + 1)\lambda^2 + (9\kappa^2 - 78\kappa + 8) - \kappa^3 + 29\kappa^2 + 29\kappa - 1. \tag{29}$$

From the inequalities

$$v(\kappa) < 0, \quad v(\kappa + 1) < 0$$

it follows according to Proposition 2.9 (ii) that

$$v(\lambda) < 0 \text{ for } \lambda \in \mathbf{J}.$$

The second inequality $\omega < 0$ in (18) is clear. So it remains to prove $Q < 0$. From (29) we get after some algebra

$$Q'(\kappa) < 0, \quad Q'(\kappa + 1) < 0$$

and – applying Proposition 2.9 (ii) again –

$$Q'(\lambda) < 0 \text{ for } \lambda \in \mathbf{J}.$$

Thus the function $8Q(\lambda)$ decreases in $\mathbf{J}$ from the initial value

$$8Q(\kappa) = -64\kappa^2 + 37\kappa - 1, \tag{30}$$

which is negative. [It is sufficient to apply Proposition 2.9 (iii) to the polynomial in (30) with $K = 1$.] Thus all inequalities in (18) are correct and Proposition 2.10 may be used.

Secondly we assume (26) and apply Proposition 2.11. The expression for $4v(\lambda)$ is identical with (28) so that (21) holds. Further

$$2Q(\lambda) = \lambda^3 - 3(\kappa + 1)\lambda^2 + (3\kappa^2 - 6\kappa + 3)\lambda - \kappa^3 + 17\kappa^2 + 17\kappa - 1.$$

From

$$Q'(\kappa) < 0, \quad Q'(\kappa + 1) < 0$$

it follows that

$$Q'(\lambda) < 0 \text{ for } \lambda \in \mathbf{J},$$

so that $Q(\lambda)$ decreases to the positive value $Q(\kappa + 1)$. Thus the first inequality in (22) is fulfilled. As far as the second is concerned we get

$$4R(\lambda) = -\lambda^2 + 2(\kappa + 1)\lambda - \kappa^2 + 11\kappa$$

with

$$R(\kappa) > 0, \quad R(\kappa + 1) > 0,$$

which is sufficient for our purposes. $4R^2 > Q$ remains. We shall write it as

$$W(\lambda) > 0 \text{ for } \lambda \in \mathbf{J}$$

with

$$W(\lambda) := \lambda^4 - 4(\kappa + 1)\lambda^3 + (6\kappa^2 - 12\kappa + 7)\lambda^2 + (-4\kappa^3 + 36\kappa^2 + 34\kappa - 6)\lambda + \kappa^4 - 20\kappa^3 + 87\kappa^2 - 34\kappa + 2.$$

Then $W''(\lambda)$ is a polynomial of the second degree fulfilling

$$W''(\kappa) < 0, \quad W''(\kappa + 1) < 0,$$

so that

$$W''(\lambda) < 0 \text{ for } \lambda \in \mathbf{J}.$$

Therefore $W'(\lambda)$ decreases in $\mathbf{J}$ to the value

$$W'(\kappa + 1) = 0,$$

being thus positive in $\mathbf{J}$. Consequently $W(\lambda)$ increases in $\mathbf{J}$ from the value

$$W(\kappa) = 2(64\kappa^2 - 20\kappa + 1), \qquad (31)$$

which is positive. [Apply Proposition 2.9 (iii) to the polynomial in (31) and put $K = 1$.] Thus all inequalities in (22) are fulfilled and Proposition 2.11 may be applied.

Finally there remains the case of (27) and Proposition 2.12. If $n = 1$ only condition (23) is to be verified. If $n = -1$ we have to check three inequalities in (24). Here we remember the relation $\kappa \le \lambda < \kappa + 1$. Thus for $\Xi = M$ our proof is completed.

For the remaining 13 selected points from Table 3 the proofs are much easier.

Lemma 2.15.

(i) If $\Xi \in \{F, J, N, O, \Gamma, \Theta, \Lambda, \Phi, \Psi\}$ then $\mathbf{36} \dot{<} \mathbf{G}_2$ (meaning that $|-\mathbf{b} + \mathbf{c}| < |\mathbf{r}|$ for any $\mathbf{r} \in \mathbf{G}_2$).

(ii) If $\Xi = \Delta$ then $\mathbf{38} \dot{<} \mathbf{G}_2$.

(iii) If $\Xi \in \{H, \Sigma\}$ then $\mathbf{39} \dot{<} \mathbf{G}_2$.

(iv) If $\Xi \in \{M, P\}$ then $\mathbf{40} \dot{<} \mathbf{G}_2$.

Proof. Let us take again $\Xi = M$. We have to prove

$$|\mathbf{40}| < |\mathbf{r}| \text{ for } \mathbf{r} \in \mathbf{G}_2,$$

that is

$$|-\mathbf{a} - \mathbf{b} + \mathbf{c}| < |m\mathbf{a} + n\mathbf{b} + p\mathbf{c}|$$

if either (25) or (26) or (27) is true. However, the last inequality is equivalent to $V > 0$ with the value $\omega = r + s - t - 1 - \kappa - \lambda$. Now we use Proposition 2.14 and Definition 2.13.

With the remaining selected points it is analogous.

Lemma 2.16. If $\Xi \in \Omega$ then $\mathbf{32} \dot{<} \mathbf{B}_2$.

Proof. We have to prove

$$(-\mathbf{a} + \mathbf{b})^2 < (m\mathbf{a} + n\mathbf{b})^2 \qquad (32)$$

for either

$$n \ge 2 \qquad (33)$$

or

$$n = 1, \quad |m| \ge 2. \qquad (34)$$

The inequality (32) means

$$W := m^2 + \kappa n^2 + \zeta mn - 1 - \kappa + \zeta > 0, \qquad (35)$$

where $\zeta := z/A$ so that $0 \le \zeta \le 1$. Because of the linearity it is sufficient to prove (35) for $\zeta = 0$ and $\zeta = 1$. In the first case we get

$$W_1 := m^2 + \kappa n^2 - 1 - \kappa > 0$$

which is true for both (33) and (34). In the second case W reads

$$W_2 := m^2 + mn + \kappa n^2 - \kappa.$$

Considering this expression as a quadratic function of m, its discriminant is $(1 - 4\kappa)n^2 + 4\kappa$.

If (33) holds it is negative because $n^2 \ge 4 > 4\kappa/(4\kappa - 1)$. Consequently W_2 has no zero points and is positive. If (34) is true the inequality $W_2 > 0$ is clear.

Lemma 2.17. If $\Xi \in \Omega$ then $\mathbf{34} \dot{<} \mathbf{R}_2$.

Proof. We have to prove

$$(-\mathbf{a} + \mathbf{c})^2 < (m\mathbf{a} + \mathbf{c})^2 \text{ for } |m| \ge 2,$$

that is

$$(m^2 - 1)A + (m + 1)y > 0 \text{ for } |m| \ge 2.$$

As in the previous lemma it is sufficient to replace y partly by 0, partly by A. One gets $m^2 - 1 > 0$, $m^2 + m > 0$, which is correct.

Remark 2.18. Lemmas 2.15, 2.16 and 2.17 were our goal in this section.

2.3. Shortest vectors in selected points

Using the results of the preceding section we can easily ascertain a few first members of the normal sequences of a-vectors, b-vectors and red c-vectors in any point Ξ of Ω. For the green c-vectors we can do it, at this moment, only for the selected points from Table 3.

The result is shown in Table 4. For example, the inequalities (11) follow from Lemma 2.16. Or, to prove the last entry in Table 4 we use Lemma 2.15 (iii) and the relations

$$\mathbf{36} \dot{<} \mathbf{39} \doteq \mathbf{40} \dot{<} \mathbf{35}, \quad \mathbf{40} \dot{<} \mathbf{37}, \quad \mathbf{40} \dot{<} \mathbf{38}$$

which follow directly from the coordinates of the point Σ.

Note that in any entry of Table 4 *the last inequality* (before the three dots ...) is *sharp*, $\dot{<}$.

All this, however, is not sufficient for constructing the minitetrahedra of L. We shall also have to know the mutual relationships between the lengths of the different kinds of vectors which appear (*i.e.* between the a-vectors and b-vectors *etc.*) These relations follow without great effort from the coordinates of the point Ξ. Sometimes, however, the inequalities

$$A \le B \le C, \quad C \le A + B \text{ and } C \ge A + B$$

must be separated into particular cases according to whether the symbol $\le$ means $<$ or $=$.

For example, if

$$\Xi = H, \quad A < B = C \qquad (36)$$

Table 4 gives

$$\mathbf{b} \doteq \mathbf{32} \dot{<} \ldots, \quad \mathbf{c} \dot{<} \mathbf{34} \dot{<} \ldots, \quad \mathbf{36} \dot{<} \mathbf{39} \dot{<} \ldots \qquad (37)$$

and this must be completed by

$$\mathbf{a} \dot{<} \mathbf{b} \doteq \mathbf{c} \doteq \mathbf{36}, \quad \mathbf{34} \doteq \mathbf{39}, \qquad (38)$$

which follows from (36). Later we shall see that the system of inequalities (37), (38) already enables us to gain all mini-tetrahedra of L and to decide which of them are mutually congruent and which are not.

In this way the system (37), (38) becomes of basic importance for us and deserves to be given a special name: we call it a *clue*. We shall record it in two ways. Where brevity is desirable (*e.g.* in tables) we write

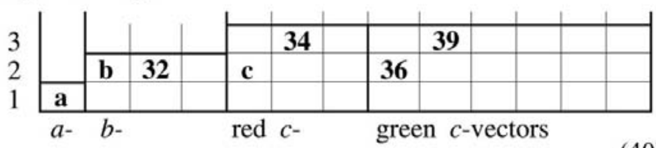
$$H(A < B = C): \mathbf{a} \mid \mathbf{b}\ \mathbf{32}\ \mathbf{c}\ \mathbf{36} \mid \mathbf{34}\ \mathbf{39}. \qquad (39)$$

The one-to-one correspondence between (37), (38) and (39) is, we hope, apparent. Do not forget, however, that (39) also covers the inequalities

$$\mathbf{32} \mathrel{\dot{<}} \ldots, \ \mathbf{34} \mathrel{\dot{<}} \ldots \ \text{ and } \mathbf{39} \mathrel{\dot{<}} \ldots.$$

Where, on the other hand, we have to deal with mutual interactions of different clues a form of the following diagram is more convenient:

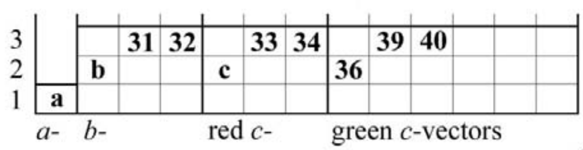
$H(A < B = C)$:



(40)

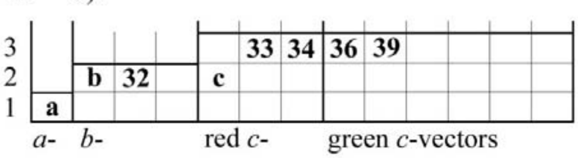
It expresses the same as the sequence (39) and as the system of inequalities (37), (38). Two further examples:

$\Sigma(A < B = C)$:



(41)

$\Phi(A < B = C)$:



(42)

The clues for Ξ ranging over all selected points from Table 3 are listed in Table 5.

2.4. Shortest vectors generally

Here the reader is reminded of the fact that General assumption 1.4 is still kept. Let $\mathbf{s}, \mathbf{t}$ be arbitrary vectors. Then the assertion

$$\text{‘the relation } \mathbf{s} \doteq \mathbf{t} \text{ (or } \mathbf{s} \mathrel{\dot{<}} \mathbf{t}\text{) holds’} \qquad (43)$$

has according to Notations 2.6 an exact definite meaning. Now we shall generalize it a little in order to simplify further formulations.

Definition 2.19. Let

$$\mathbf{s} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}, \quad \mathbf{t} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c} \qquad (44)$$

be lattice vectors of L, let j be an integer and $\Xi_j \in \Omega$. We say that

$$\text{‘the relation } \mathbf{s} \doteq \mathbf{t} \text{ (or } \mathbf{s} \mathrel{\dot{<}} \mathbf{t}\text{) holds in the point } \Xi_j\text{’} \qquad (45)$$

if

$$|m\mathbf{a}_j + n\mathbf{b}_j + p\mathbf{c}_j| = |u\mathbf{a}_j + v\mathbf{b}_j + w\mathbf{c}_j| \qquad (46)$$

$$(\text{or } |m\mathbf{a}_j + n\mathbf{b}_j + p\mathbf{c}_j| < |u\mathbf{a}_j + v\mathbf{b}_j + w\mathbf{c}_j|) \qquad (47)$$

is true for some linearly independent vectors $\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j$ (which need not be lattice vectors of L) fulfilling

$$|\mathbf{a}_j| = |\mathbf{a}|, \quad |\mathbf{b}_j| = |\mathbf{b}|, \quad |\mathbf{c}_j| = |\mathbf{c}|, \qquad (48)$$

$$[\xi_j, \eta_j, \zeta_j] = \Xi_j, \qquad (49)$$

where

$$\xi_j = 2\mathbf{b}_j \cdot \mathbf{c}_j, \quad \eta_j = 2\mathbf{a}_j \cdot \mathbf{c}_j, \quad \zeta_j = 2\mathbf{a}_j \cdot \mathbf{b}_j. \qquad (50)$$

[Thus the assertion (43) is equivalent to the assertion (45) for $\Xi_j = \Xi$.]

Lemma 2.20. Let $\Xi_1, \Xi_2 \in \Omega$, let Ξ_0 lie on the open straight segment with the end points Ξ_1, Ξ_2. Let (44) be arbitrary lattice vectors of L. Then the following is true:

(i) if $\mathbf{s} \doteq \mathbf{t}$ in the point Ξ_1 and $\mathbf{s} \doteq \mathbf{t}$ in the point Ξ_2 then $\mathbf{s} \doteq \mathbf{t}$ in the point Ξ_0;

(ii) if ($\mathbf{s} \mathrel{\dot{<}} \mathbf{t}$ or $\mathbf{s} \doteq \mathbf{t}$) in the point Ξ_1 and $\mathbf{s} \mathrel{\dot{<}} \mathbf{t}$ in the point Ξ_2 then $\mathbf{s} \mathrel{\dot{<}} \mathbf{t}$ in the point Ξ_0.

Proof. Let $\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j$ ($j = 0, 1, 2$) be linearly independent vectors fulfilling (48), (49) and (50). Denote

$$F(\xi, \eta, \zeta) := (np - vw)\xi + (mp - uw)\eta + (mn - uv)\zeta + (m^2 - u^2)\mathbf{a}^2 + (n^2 - v^2)\mathbf{b}^2 + (p^2 - w^2)\mathbf{c}^2.$$

From (46) and (47) it follows that the assertion

$$\text{‘}\mathbf{s} \doteq \mathbf{t} \text{ (or } \mathbf{s} \mathrel{\dot{<}} \mathbf{t}\text{) holds in the point } \Xi_j\text{’}$$

means the same as

$$\text{‘}F(\Xi_j) = 0 \ (\text{or } F(\Xi_j) < 0)\text{’}.$$

The implications (i), (ii) now read

$$\text{‘if } F(\Xi_1) = 0 \text{ and } F(\Xi_2) = 0 \text{ then } F(\Xi_0) = 0\text{’} \qquad (51)$$

and

$$\text{‘if } F(\Xi_1) \le 0 \text{ and } F(\Xi_2) < 0 \text{ then } F(\Xi_0) < 0\text{’}. \qquad (52)$$

First let us suppose that at least one of the numbers

$$np - vw, \quad mp - uw, \quad mn - uv \qquad (53)$$

is different from zero. Then $F = 0$ is a plane dividing the space into two open subspaces, $F < 0$ and $F > 0$, and the implications (51) and (52) are clear.

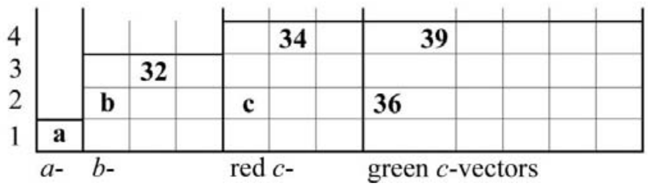
Secondly, let all three numbers (53) be equal to zero. Then

$$F(\Xi_1) = F(\Xi_2) = F(\Xi_3)$$

and the implications (51) and (52) are true again.

Examples 2.21. From the clues (40) and (41) we get immediately[5]

$H\Sigma(A < B = C)$:

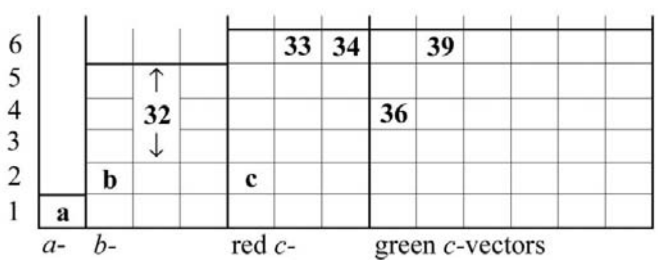


($H\Sigma$ means the *open* straight segment with the end points H, Σ. Similarly for the triangles and tetrahedra.) Similarly we get

$$H\Phi(A < B = C) : \mathbf{a} \mid \mathbf{b}\ \mathbf{32}\ \mathbf{c} \mid \mathbf{36}\ \mid \mathbf{34}\ \mathbf{39}.$$

With $\Sigma\Phi$ it is somewhat more complicated because of $\mathbf{36} \dot{<} \mathbf{32}$ in the point Σ and $\mathbf{32} \dot{<} \mathbf{36}$ in the point Φ. Here Lemma 2.20 fails. However, because of linearity there exists a point, say D, on the straight segment $\Sigma\Phi$ in which $\mathbf{32} \doteq \mathbf{36}$. It would be possible to add this point to the selected points and construct now three diagrams, namely for the point D and the straight segments $D\Sigma$ and $D\Phi$. Fortunately, for our purposes (*i.e.* the uniqueness and ambiguity) this is not necessary and we can deal with the straight segment $\Sigma\Phi$ as a whole. Both forms of the clue can be modified, *e.g.* in this transparent way:

$\Sigma\Phi(A < B = C)$:



and

$$\Sigma\Phi(A < B = C) : \mathbf{a} \mid \mathbf{b}\ \ \mathbf{c} \mid \mathbf{32} \sim \mathbf{36} \mid \mathbf{33}\ \ \mathbf{34}\ \ \mathbf{39}.$$

Taking the open triangle $H\Sigma\Phi$ as a set of straight segments HD we get the clues in all points of this triangle. Finally combining these clues with the clue in the selected point Θ (see Table 5) we obtain the clue in any point of the open tetrahedron $H\Theta\Sigma\Phi$.

In this way we can ascertain the clues in all points of Ω. However, they are not systematically tabulated in this paper.

2.5. Distribution of the *a*-, *b*- and *c*-vectors along the edges of a lattice tetrahedron

Apparently this distribution cannot be arbitrary, since the sum of vectors lying on the border of any side of a lattice tetrahedron can be made zero by changing the direction of at most one of these vectors. A detailed analysis leads to the following lemma:

Table 4
The few first members of the normal sequences.

Type of vector	Condition for the point Ξ	First members of the normal sequences
a-vectors		$\mathbf{a} \dot{<} \ldots$
b-vectors	$0 = z$	$\mathbf{b} \dot{<} \mathbf{31} \doteq \mathbf{32} \dot{<} \ldots$
	$0 < z < A$	$\mathbf{b} \dot{<} \mathbf{32} \dot{<} \ldots$
	$z = A$	$\mathbf{b} \doteq \mathbf{32} \dot{<} \ldots$
Red *c*-vectors	$0 = y$	$\mathbf{c} \dot{<} \mathbf{33} \doteq \mathbf{34} \dot{<} \ldots$
	$0 < y < A$	$\mathbf{c} \dot{<} \mathbf{34} \dot{<} \ldots$
	$y = A$	$\mathbf{c} \doteq \mathbf{34} \dot{<} \ldots$
Green *c*-vectors	F†	$\mathbf{36} \doteq \mathbf{38} \dot{<} \ldots$
	H	$\mathbf{36} \dot{<} \mathbf{39} \dot{<} \ldots$
	J	$\mathbf{36} \dot{<} \ldots$
	M, P	$\mathbf{36} \dot{<} \mathbf{40} \dot{<} \ldots$
	N, Ψ	$\mathbf{36} \doteq \mathbf{40} \dot{<} \ldots$
	O	$\mathbf{35} \doteq \mathbf{36} \dot{<} \ldots$
	Γ	$\mathbf{35} \doteq \mathbf{36} \doteq \mathbf{38} \doteq \mathbf{39} \dot{<} \ldots$
	Δ	$\mathbf{38} \dot{<} \ldots$
	Θ, Φ	$\mathbf{36} \doteq \mathbf{39} \dot{<} \ldots$
	Λ	$\mathbf{35} \doteq \mathbf{36} \doteq \mathbf{38} \doteq \mathbf{40} \dot{<} \ldots$
	Σ	$\mathbf{36} \dot{<} \mathbf{39} \doteq \mathbf{40} \dot{<} \ldots$

† Meaning $\Xi = F$.

Lemma 2.22. All possible distributions of the *a*-, *b*- and *c*-vectors along the edges of a lattice tetrahedron are shown in Fig. 7. They are called *types* and are denoted

$$\mathbf{T}_1, \ldots, \mathbf{T}_7. \qquad (54)$$

However, this is not enough. We also have to distinguish between the red *c*-vectors and the green *c*-vectors. Fortunately, we need not know the concrete edges in which these vectors lie; it is sufficient to know only their number. And this is limited by the following proposition:

Proposition 2.23. In the edges of an arbitrary lattice tetrahedron there lie at most two red *c*-vectors.

Proof. Let us suppose that there are three such *c*-vectors. Then they either meet in a vertex or form a 'chain'. In neither of these cases would the supposed 'tetrahedron' be a three-dimensional body.

Applying this result to the seven types (54) we get 21 *subtypes* denoted

$$\mathbf{T}_{i0}, \mathbf{T}_{i1}, \mathbf{T}_{i2}, \quad (i = 1, \ldots, 7) \qquad (55)$$

the second subscript indicating the number of the red *c*-vectors. For expressing these subtypes explicitly we introduce a special notation.

Notation 2.24 (the standard description of vectors lying in the edges of a lattice tetrahedron). Let T be an arbitrary lattice tetrahedron.

(i) If there is an *a*-vector lying in an edge of T we denote it $\mathbf{a}'_1$.

[5] Remember that diagram (40) also contains, for example, the inequality $\mathbf{32} \dot{<} \mathbf{31}$ in the point H which with $\mathbf{32} \doteq \mathbf{31}$ in Σ gives $\mathbf{32} \dot{<} \mathbf{31}$ in $H\Sigma$.

Table 5
Clues in the selected points.

To make the table as concise as possible the following conventions are made: If $A < B$ the symbol ':' is replaced by |. Otherwise the symbol ':' is deleted from the table. Analogous conventions are made for $B < C$, ';', and $A + B < C$, ','.

Point	Condition	Clue
F		**a** : **b** **32** ; **c** **34** $\mid$ **36** **38**
H		**a** : **b** **32** ; **c** **36** $\mid$ **34** **39**
J		**a** : **b** **32** ; **c** **34** **36**
M		**a** : **b** ; **c** **34** **36** $\mid$ **32** **40**†
N		**a** : **b** ; **c** **34** $\mid$ **31** **32** **36** **40**†
O	$C < A + B$	**a** : **b** ; **c** $\mid$ **31** **32** ; **33** **34** : **35** **36**
O	$C \geq A + B$	**a** : **b** $\mid$ **31** **32** , **c** $\mid$ **33** **34** : **35** **36**
P		**a** : **b** $\mid$ **c** **36** $\mid$ **31** **32** **34** **40**†
Γ		**a** : **b** **32** ; **c** $\mid$ **33** **34** : **35** **36** **38** **39**
Δ		**a** : **b** **32** ; **c** **34** : **38**
Θ		**a** : **b** **32** ; **c** **36** **39** $\mid$ **33** **34**
Λ	$A = C$	**a** **b** **c** **34** $\mid$ **31** **32** **35** **36** **38** **40**
Λ	$A < C < A + B$	**a** : **b** ; **c** **34** $\mid$ **31** **32** $\mid$ **35** **36** **38** **40**
Λ	$C \geq A + B$	**a** : **b** $\mid$ **31** **32** , **c** **34** $\mid$ **35** **36** **38** **40**
Σ	$C < A + B$	**a** : **b** ; **c** **36** $\mid$ **31** **32** ; **33** **34** **39** **40**
Σ	$C \geq A + B$	**a** : **b** $\mid$ **31** **32** , **c** **36** $\mid$ **33** **34** **39** **40**
Φ		**a** $\mid$ **b** **32** ; **c** $\mid$ **33** **34** **36** **39**†
Ψ	$C < A + B$	**a** : **b** ; **c** **34** **36** **40** $\mid$ **31** **32**
Ψ	$C \geq A + B$	**a** : **b** $\mid$ **31** **32** , **c** **34** **36** **40**

† Remember the restrictions for the coordinates of this point in Table 3.

(ii) If there are b-vectors lying in the edges of T we denote them

$$\mathbf{b}_1', \ldots, \mathbf{b}_m' \quad (1 \leq m \leq 3)$$

in such a way that

$$|\mathbf{b}_1'| \leq \ldots \leq |\mathbf{b}_m'|.$$

(iii) Similarly we denote by

$$\mathbf{r}_1', \ldots, \mathbf{r}_p' \quad (|\mathbf{r}_1'| \leq \ldots \leq |\mathbf{r}_p'|, \quad 1 \leq p \leq 2)$$

the red c-vectors lying in the edges of T and

$$\mathbf{g}_1', \ldots, \mathbf{g}_q' \quad (|\mathbf{g}_1'| \leq \ldots \leq |\mathbf{g}_q'|, \quad 1 \leq q \leq 6)$$

the green c-vectors from the edges of T.

Now we are prepared to describe the subtypes (55) explicitly. This is done in Table 6.

In Table 2 we introduced an abbreviated notation of some vectors by means of bold integers. Now the time has come to do the same thing for some abstract tetrahedra (this time, however, in bold italics). This is realized in Table 7.

The next notion has appeared already in Intermezzo 2.8.

Definition 2.25. We say that T is an *auxiliary* tetrahedron if in its edges there lie the vectors **a**, **b**, **c**. The term extends to the abstract tetrahedron containing T.

Lemma 2.26. In the lattice L there exist exactly 16 auxiliary abstract tetrahedra, namely ***51***, ..., ***66***.

Proof. The vectors **a**, **b**, **c** either meet in one vertex of T or form a 'chain' which has in its middle either the vector **a**, or **b**, or **c**. Then we change the direction of these vectors in all possible ways.

2.6. The first minitetrahedron

In this section we shall show how to ascertain whether a given lattice tetrahedron T_0 is a minitetrahedron of L. We shall explain it in the following example.

Example 2.27. Let

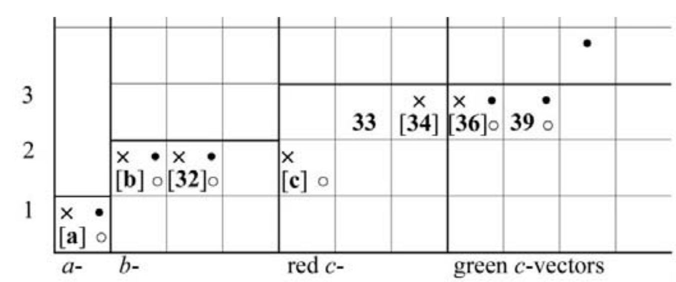
$$\Xi = \Phi, \quad A < B = C. \tag{56}$$

The considerations in Intermezzo 2.8 or clue (42) together with Table 7 suggest that $T_0 \in \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ might be a good guess. To prove it we have to show

$$\sigma(T) \geq \sigma(T_0) \tag{57}$$

for any lattice tetrahedron T regardless of its subtype $\mathbf{T}_{ij}$.

We modify the diagram (42) to (58) by placing the vectors lying in the edges of T_0 in square brackets:



(58)

Suppose first that the tetrahedron T is of the subtype $\mathbf{T}_{10}$. Then in the standard description given by Notation 2.24

$$|\mathbf{a}_1'| \geq |\mathbf{a}|, \quad |\mathbf{b}_1'| \geq |\mathbf{b}|, \quad |\mathbf{b}_2'| \geq |\mathbf{32}|, \tag{59}$$

Table 6
Description of the 21 subtypes $\mathbf{T}_{ij}$.

	$\mathbf{a}'_1$	$\mathbf{b}'_1$	$\mathbf{b}'_2$	$\mathbf{b}'_3$	$\mathbf{r}'_1$	$\mathbf{r}'_2$	$\mathbf{g}'_1$	$\mathbf{g}'_2$	$\mathbf{g}'_3$	$\mathbf{g}'_4$	$\mathbf{g}'_5$	$\mathbf{g}'_6$
$\mathbf{T}_{10}$	•	•	•				•	•	•			
$\mathbf{T}_{11}$	•	•	•		•		•	•				
$\mathbf{T}_{12}$	•	•	•		•	•	•					
$\mathbf{T}_{20}$		•	•	•			•	•	•			
$\mathbf{T}_{21}$		•	•	•	•		•	•				
$\mathbf{T}_{22}$		•	•	•	•	•	•					
$\mathbf{T}_{30}$	•	•					•	•	•	•		
$\mathbf{T}_{31}$	•	•			•		•	•	•			
$\mathbf{T}_{32}$	•	•			•	•	•	•				
$\mathbf{T}_{40}$		•	•				•	•	•	•		
$\mathbf{T}_{41}$		•	•		•		•	•	•			
$\mathbf{T}_{42}$		•	•		•	•	•	•				
$\mathbf{T}_{50}$	•						•	•	•	•	•	
$\mathbf{T}_{51}$	•				•		•	•	•	•		
$\mathbf{T}_{52}$	•				•	•	•	•	•			
$\mathbf{T}_{60}$		•					•	•	•	•	•	
$\mathbf{T}_{61}$		•			•		•	•	•	•		
$\mathbf{T}_{62}$		•			•	•	•	•	•			
$\mathbf{T}_{70}$							•	•	•	•	•	•
$\mathbf{T}_{71}$					•		•	•	•	•	•	
$\mathbf{T}_{72}$					•	•	•	•	•	•		

$$|\mathbf{g}'_1| \geq |\mathbf{36}|, \quad |\mathbf{g}'_2| \geq |\mathbf{39}| = |\mathbf{34}|, \quad |\mathbf{g}'_3| > |\mathbf{36}| > |\mathbf{c}|. \qquad (60)$$

[The least 'favourable' alternative (see Remarks 2.5) is indicated in diagram (58) by filled circles, •.] From (59) and (60) it follows that $\sigma(T) > \sigma(T_0)$, which we shall write in short as $\mathbf{T}_{10} > 0$.

Secondly let T be of the subtype $\mathbf{T}_{11}$. Then the inequalities (59) remain whereas (60) changes to

$$|\mathbf{r}'_1| \geq |\mathbf{c}|, \quad |\mathbf{g}'_1| \geq |\mathbf{36}|, \quad |\mathbf{g}'_2| \geq |\mathbf{39}| = |\mathbf{34}|, \qquad (61)$$

which leads to (57); in short $\mathbf{T}_{11} \geq 0$. [The 'worst' possibility is indicated in (58) by open circles, ○.]

Table 7
Brief notation of some abstract tetrahedra with vectors lying in their edges.

Symbol	Tetrahedron	Vectors in its edges					
51	$\langle\mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$	**a**	**b**	**c**	**32**	**34**	**36**
52	$\langle\mathbf{a}, \mathbf{b}, -\mathbf{c}\rangle$	**a**	**b**	**c**	**32**	**33**	**35**
53	$\langle\mathbf{a}, -\mathbf{b}, \mathbf{c}\rangle$	**a**	**b**	**c**	**31**	**34**	**35**
54	$\langle-\mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$	**a**	**b**	**c**	**31**	**33**	**36**
55	$\langle\mathbf{a}, \mathbf{b}, \mathbf{a} + \mathbf{c}\rangle$	**a**	**b**	**c**	**32**	**33**	**39**
56	$\langle\mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{c}\rangle$	**a**	**b**	**c**	**32**	**34**	**38**
57	$\langle\mathbf{a}, -\mathbf{b}, \mathbf{a} + \mathbf{c}\rangle$	**a**	**b**	**c**	**31**	**33**	**37**
58	$\langle\mathbf{a}, \mathbf{b}, \mathbf{b} + \mathbf{c}\rangle$	**a**	**b**	**c**	**32**	**35**	**38**
59	$\langle\mathbf{a}, \mathbf{b}, \mathbf{b} - \mathbf{c}\rangle$	**a**	**b**	**c**	**32**	**36**	**39**
60	$\langle-\mathbf{a}, \mathbf{b}, \mathbf{b} + \mathbf{c}\rangle$	**a**	**b**	**c**	**31**	**35**	**37**
61	$\langle\mathbf{a}, \mathbf{c}, \mathbf{a} + \mathbf{b}\rangle$	**a**	**b**	**c**	**31**	**34**	**40**
62	$\langle\mathbf{a}, \mathbf{c}, \mathbf{b} + \mathbf{c}\rangle$	**a**	**b**	**c**	**34**	**35**	**38**
63	$\langle\mathbf{a}, \mathbf{c}, -\mathbf{b} + \mathbf{c}\rangle$	**a**	**b**	**c**	**34**	**36**	**40**
64	$\langle-\mathbf{a}, \mathbf{c}, \mathbf{b} + \mathbf{c}\rangle$	**a**	**b**	**c**	**33**	**35**	**37**
65	$\langle\mathbf{b}, \mathbf{c}, \mathbf{a} + \mathbf{b}\rangle$	**a**	**b**	**c**	**31**	**36**	**40**
66	$\langle\mathbf{b}, \mathbf{c}, \mathbf{a} + \mathbf{c}\rangle$	**a**	**b**	**c**	**33**	**36**	**39**
67	$\langle\mathbf{a}, \mathbf{b}, -\mathbf{a} - \mathbf{b} + \mathbf{c}\rangle$	**a**	**b**	**32**	**34**	**36**	**40**

Finally, if T is of the subtype $\mathbf{T}_{12}$ the inequalities (59) are kept again and instead of (60) we have now

$$|\mathbf{r}'_1| \geq |\mathbf{c}|, \quad |\mathbf{r}'_2| \geq |\mathbf{34}|, \quad |\mathbf{g}'_1| \geq |\mathbf{36}|$$

[see the crosses × in (58)]. The inequality (57) is true. Summarizing we can write in short for the type $\mathbf{T}_1$

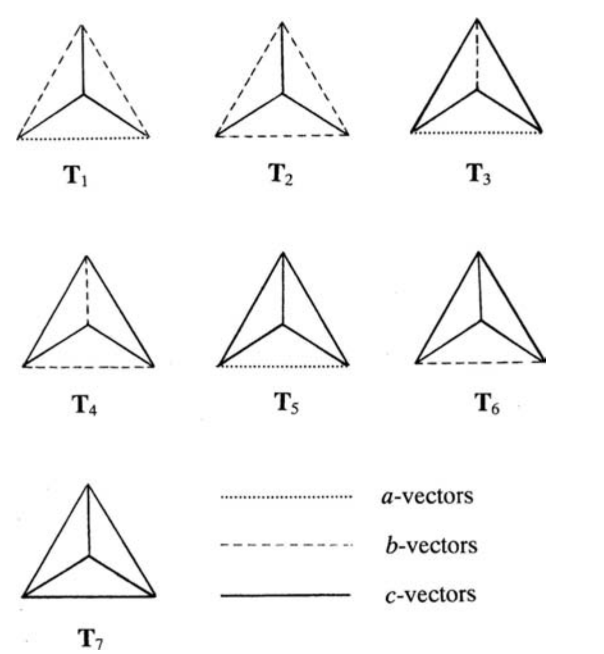


Figure 7
Definition of the seven types of lattice tetrahedra according to Lemma 2.22.

$$\mathbf{T}_{10} > 0, \quad \mathbf{T}_{11} \geq 0, \quad \mathbf{T}_{12} \geq 0. \tag{62}$$

In the same way all remaining types $\mathbf{T}_i$ ($2 \leq i \leq 7$) can be treated. It turns out that in all these cases we get sharp inequalities

$$\mathbf{T}_{ij} > 0 \text{ for } 2 \leq i \leq 7, \; 0 \leq j \leq 2. \tag{63}$$

Thus we are justified in making the following proposition:

Proposition 2.28. On the condition (56) the abstract tetrahedron $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a minitetrahedron of L.

2.7. All minitetrahedra

In this section we continue Example 2.27. We know already that $T_0 \in \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ is a minitetrahedron of L. Now let T' be an arbitrary minitetrahedron in a standard description so that

$$\sigma(T') = \sigma(T_0). \tag{64}$$

From (62) and (63) it follows that T' must be either of the subtype $\mathbf{T}_{11}$ or $\mathbf{T}_{12}$.

First let T' be of the subtype $\mathbf{T}_{11}$. The relations (64), (59) and (61) give

$$(|\mathbf{a}_1'| - |\mathbf{a}|) + (|\mathbf{b}_1'| - |\mathbf{b}|) + (\mathbf{b}_2'| - |\mathbf{32}|) + (|\mathbf{r}_1'| - |\mathbf{c}|) + (|\mathbf{g}_1'| - |\mathbf{36}|) + (|\mathbf{g}_2'| - |\mathbf{34}|) = 0,$$

where any expression in parentheses is non-negative. Thus it is actually zero. Hence

$$\mathbf{a}_1' = \mathbf{a}, \quad \{\mathbf{b}_1', \mathbf{b}_2'\} = \{\mathbf{b}, \mathbf{32}\}, \tag{65}$$

$$\mathbf{r}_1' = \mathbf{c}, \quad \{\mathbf{g}_1', \mathbf{g}_2'\} = \{\mathbf{36}, \mathbf{39}\}, \tag{66}$$

according to Definition 2.4. Thus T' must have in its edges the vectors

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{32}, \; \mathbf{c}, \; \mathbf{36}, \; \mathbf{39}$$

which admits only the abstract tetrahedron ***59*** (see Lemma 2.26 and Table 7).

Secondly let T' be of the subtype $\mathbf{T}_{12}$. This time we get in a similar way (65) and

$$\mathbf{r}_1' = \mathbf{c}, \quad \mathbf{r}_2' \in \{\mathbf{33}, \mathbf{34}\}, \quad \mathbf{g}_1' \in \{\mathbf{36}, \mathbf{39}\}.$$

Thus in the edges of T' there must lie either the vectors

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{32}, \; \mathbf{c}, \; \mathbf{33}, \; \mathbf{36} \tag{67}$$

or

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{32}, \; \mathbf{c}, \; \mathbf{33}, \; \mathbf{39} \tag{68}$$

or

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{32}, \; \mathbf{c}, \; \mathbf{34}, \; \mathbf{36} \tag{69}$$

or

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{32}, \; \mathbf{c}, \; \mathbf{34}, \; \mathbf{39}. \tag{70}$$

In any case T' is an auxiliary tetrahedron and we can use Lemma 2.26 and Table 7. They show that the vectors (68) and (69) lead to the abstract tetrahedra ***55*** and ***51*** whereas (67) and (70) can lie in edges of no tetrahedron. Summarizing, we get the following proposition:

Proposition 2.29. On the conditions (56) the lattice L has exactly three abstract minitetrahedra, namely

$$\mathbf{\mathit{51}} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle, \quad \mathbf{\mathit{55}} = \langle \mathbf{a}, \mathbf{b}, \mathbf{a} + \mathbf{c} \rangle, \quad \mathbf{\mathit{59}} = \langle \mathbf{a}, \mathbf{b}, \mathbf{b} - \mathbf{c} \rangle. \tag{71}$$

2.8. Uniqueness and ambiguity

Here we conclude Example 2.27. The explicit distribution of vectors lying in the edges of the tetrahedra (71) is demonstrated in the first row of Fig. 8. Substituting these vectors by the numbers of their 'levels' in diagram (58) (which is done in the second row of Fig. 8) we can immediately see which of the tetrahedra (71) are mutually congruent and which are not. Thus we can conclude:

Proposition 2.30. On the conditions (56) the abstract tetrahedra $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$ and $\langle \mathbf{a}, \mathbf{b}, \mathbf{a} + \mathbf{c} \rangle$ are congruent, but are not congruent with $\langle \mathbf{a}, \mathbf{b}, \mathbf{b} - \mathbf{c} \rangle$.

2.9. Completing the main proof

Repeating the procedure described in §§2.6, 2.7 and 2.8 we can – for any $\Xi \in \Omega$ – ascertain not only all minitetrahedra of L but also their potential mutual congruence. [The analogy with Example 2.27 is close. A smaller complication occurs only in the case

$$\Psi(C = A + B) : \mathbf{a} : \mathbf{b} \mid \mathbf{c} \; \mathbf{31} \; \mathbf{32} \; \mathbf{34} \; \mathbf{36} \; \mathbf{40}.$$

Here, between the sequences of vectors corresponding to the sequences (67), (68), (69) and (70) in Example 2.27 there appear two sequences, namely

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{31}, \; \mathbf{34}, \; \mathbf{36}, \; \mathbf{40} \tag{72}$$

and

$$\mathbf{a}, \; \mathbf{b}, \; \mathbf{32}, \; \mathbf{34}, \; \mathbf{36}, \; \mathbf{40}, \tag{73}$$

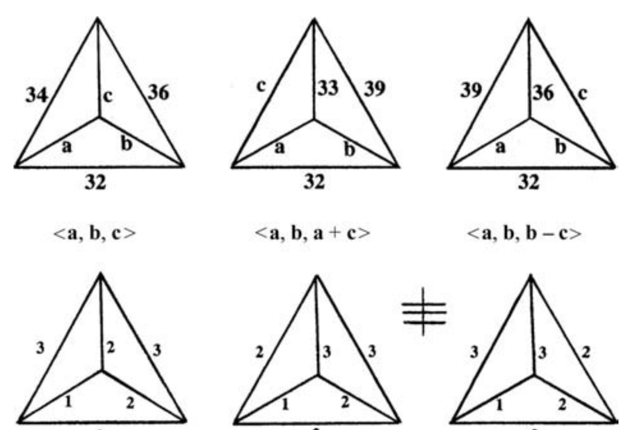


Figure 8
Congruence of minitetrahedra in Example 2.27.

Table 8
The set Ω_a of all ambiguity points with the corresponding minitetrahedra.

Points	Condition	Minitetrahedra
M	$B < C < A + B$	***51*** $\neq$ ***63***
N	$A < C < A + B$	***51 61*** $\neq$ ***63***
P	$B < C < A + B$	***51 65*** $\neq$ ***63***
Φ	$A < B$	***51 55*** $\neq$ ***59***
Ψ	$C = A + B$	***51 61 65 67*** $\neq$ ***63***
$H\Phi$	$A < B$	***51*** $\neq$ ***59***
MN	$A < C < A + B$	***51*** $\neq$ ***63***
MP	$B < C < A + B$	***51*** $\neq$ ***63***
NP	$B < C < A + B$	***51*** $\neq$ ***63***
$N\Sigma$	$A < B = C$	***51*** $\neq$ ***63***
$\Gamma\Lambda$		***51*** $\neq$ ***56***
$\Gamma\Sigma$	$A = B$	***51*** $\neq$ ***59***
$\Lambda\Sigma$	$A = C$	***51*** $\neq$ ***63***
$\Sigma\Phi$	$A < B$	***51*** $\neq$ ***59***
$F\Gamma\Lambda$		***51*** $\neq$ ***56***
$H\Gamma\Sigma$	$A = B$	***51*** $\neq$ ***59***
$H\Sigma\Phi$	$A < B$	***51*** $\neq$ ***59***
MNP	$B < C < A + B$	***51*** $\neq$ ***63***
$MN\Sigma$	$A < B = C$	***51*** $\neq$ ***63***
$M\Lambda\Sigma$	$A = C$	***51*** $\neq$ ***63***

lacking the vector **c**. Therefore we cannot use Lemma 2.26 and Table 7 as we did in §2.7. It is left to the reader to prove directly that (73) leads to $\boldsymbol{67} = \langle \mathbf{a}, \mathbf{b}, -\mathbf{a} - \mathbf{b} + \mathbf{c} \rangle$ while (72) describes no tetrahedron.] Consequently we can divide the set Ω into two classes, Ω_u when the minitetrahedron of L is unique and Ω_a when it is ambiguous.

The set Ω_a is described with all necessary details in Table 8. From this table the proofs of most theorems and propositions in §1.3 unfold. The set Ω_u is, of course, a complement of Ω_a to Ω but the minitetrahedra belonging to the particular points $\Xi \in \Omega_u$ are not given explicitly in this paper.

The proof of Theorem 1.5 follows immediately from Table 8. Before starting the proof of Theorem 1.6 it is perhaps best to illustrate the set Ω_a. This is done – although with Ω_a divided into three parts – in Figs. 3, 4 and 5. From these figures the conditions (i), (ii), (iii) in Theorem 1.6 follow. More details are in Table 1, which is partly a consequence of Table 8, partly must be completed directly.

Proposition 1.7 follows from Table 8; Proposition 1.8 needs some complementary work concerning the points from Ω_u.

Theorem 1.9: Notice the way in which in §2.7 the minitetrahedra of L are generated from the first already known minitetrahedron.

As far as Theorem 1.10 is concerned, it is sufficient to realize that the inequality (57) in §2.6 was being proved by proving inequalities between separate *pairs* of terms and that $p < q$ means the same as $p^\tau < q^\tau$ for all positive real numbers p, q, τ.

For proving Theorem 1.12 the material from Gruber (1978) is very useful.

Thus all assertions in §1.3 may be considered proved.

3. Conclusion and outlook

The author hopes to find a successor who will complete the whole problem by solving it also in the negative region

$$a + b + c = \min, \quad \mathbf{b} \cdot \mathbf{c} < 0, \; \mathbf{a} \cdot \mathbf{c} < 0, \; \mathbf{a} \cdot \mathbf{b} < 0.$$

A few preliminary probes show that surprising results may be expected. For example, the set Ω^- (defined in an analogous way to Ω in Notation 2.1) is divided into regions (indicating particular minitetrahedra) quite differently to Ω in Fig. 6. Its division resembles more a weathercock with a threefold axis going through Ω^-. This suggests a possibility of thrice ambiguous minitetrahedra. That they really occur confirms the following example:

Example 3.1. Let the basis **a**, **b**, **c** of the lattice L fulfil

$$\begin{pmatrix} \mathbf{a}^2 & \mathbf{b}^2 & \mathbf{c}^2 \\ 2\mathbf{b}\cdot\mathbf{c} & 2\mathbf{a}\cdot\mathbf{c} & 2\mathbf{a}\cdot\mathbf{b} \end{pmatrix} = \begin{pmatrix} 6 & 9 & 12 \\ -7 & -4 & -1 \end{pmatrix}.$$

Then the minitetrahedron of the lattice L is thrice ambiguous.

Hint. Having proved that any of the vectors

$$\pm\mathbf{a}, \; \pm\mathbf{b}, \pm\mathbf{c}, \; \pm\mathbf{31}, \; \pm\mathbf{33}, \; \pm\mathbf{35}, \; \pm\mathbf{37}$$

is shorter than any of the remaining nonzero lattice vectors of L, we construct the abstract tetrahedra

$$\langle \mathbf{a}, -\mathbf{b}, \mathbf{a} + \mathbf{c} \rangle, \quad \langle -\mathbf{a}, \mathbf{b}, \mathbf{b} + \mathbf{c} \rangle, \quad \langle -\mathbf{a}, \mathbf{c}, \mathbf{b} + \mathbf{c} \rangle.$$

The author expresses his gratitude to V. Petříček, C. Novák and J. Bergerová from the Institute of Physics Academy of Sciences of the Czech Republic for the final realization of the paper.

References

Eisenstein, G. (1851). *J. Reine Angew. Math. (Crelle)*, **41**, 141–190.
Gruber, B. (1973). *Acta Cryst.* A**29**, 433–440.
Gruber, B. (1978). *Comments on Bravais Lattices.* Preprint. Unpublished.
Gruber, B. (1997). *Acta Cryst.* A**53**, 505–521.
International Tables for Crystallography (2002). Vol. A, chs. 9.2, 9.3. Dordrecht: Kluwer Academic Publishers.
Křivý, I. & Gruber, B. (1976). *Acta Cryst.* A**32**, 297–298.
Niggli, P. (1928). *Handbuch der Experimentalphysik*, Vol. 7, Part 1, ch. 4, §1. Leipzig: Akademische Verlagsgesellschaft.